J. Fluid Mech. (2003), *vol.* 480, *pp.* 61–63.
DOI: 10.1017/S0022112003003720 Printed in the United Kingdom

Minimum dissipation rate flow with given flux

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(Received 25 November 2002 and in revised form 19 December 2002)

Flow of a viscous incompressible fluid through a conduit with rigid walls is considered, with the flux given at the entrance and exit. The velocity distribution which minimizes the rate of energy dissipation is characterized. It is a Stokes flow with the surface stress equal to a constant pressure at the entrance and another constant pressure at the exit.

1. Introduction

For given flux, what velocity distribution $u(x)$ minimizes the rate of energy dissipation $D[u]$, in a viscous incompressible fluid flowing through a conduit? This question leads to a natural modification of the result of Helmholtz and Korteweg, which applies when $u(x)$ is given at the entrance S_- and exit S_+. Then the solution is the Stokes flow which has those values at $S_\pm$ and vanishes on the rigid conduit surface S_1 (Batchelor 1967, p. 228).

To determine $u(x)$ when only the flux Q is known, we begin with the definition

$$D[u] = \int_V 2\mu (e_{ij}[u])^2 \, \mathrm{d}V. \tag{1.1}$$

Here V is the fluid domain, bounded by S_+, S_- and S_1, μ is the coefficient of viscosity, $e_{ij}[u] = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the strain rate, u_i is the ith component of u, and $u_{i,j} = \partial u_i / \partial x_j$. Terms with repeated indices are summed over the three values 1, 2, 3 of those indices. The stress tensor corresponding to u is $\tau_{ij}[u] = 2\mu e_{ij}[u] - p(x)\delta_{ij}$ where $p(x)$ is the pressure.

An admissible velocity field is a smooth flow $u(x)$ defined for x in V, satisfying the conditions

$$u_{i,i}(x) = 0, \qquad x \text{ in } V, \tag{1.2a}$$

$$u_i(x) = 0, \qquad x \text{ in } S_1, \tag{1.2b}$$

$$\int_{S_+} u_i n_i \, \mathrm{d}S = -\int_{S_-} u_i n_i \, \mathrm{d}S = Q. \tag{1.2c}$$

In (1.2*c*) n_i is the ith component of the unit normal to S_+ or S_- pointing out of V.

The dissipation minimizing u is characterized by the following theorem.

Among all admissible velocity fields, $D[u]$ is minimized by a unique $u(x)$ which satisfies the equations

$$\tau_{ij,j}[u(x)] = 0, \qquad x \text{ in } V, \tag{1.3a}$$

$$\tau_{ij}[u(x)]n_j(x) = -p_+ n_i(x), \qquad x \text{ in } S_+, \tag{1.3b}$$

$$= -p_- n_i(x), \qquad x \text{ in } S_-. \tag{1.3c}$$

In (1.3*c*) p_+ and p_- are constants.

2. Proof of theorem

Let $u(x)$ be an admissible velocity field satisfying the Stokes equation (1.3*a*). Any other admissible flow can be written as $u + v$, where v satisfies (1.2*a*), (1.2*b*) and

$$\int_{S_+} v_i n_i \, \mathrm{d}S = -\int_{S_-} v_i n_i \, \mathrm{d}S = 0. \tag{2.1}$$

Substituting $u + v$ into (1.1) yields

$$D[u+v] = D[u] + D[v] + 2\int_V 2\mu e_{ij}[u]e_{ij}[v] \, \mathrm{d}V. \tag{2.2}$$

The definition of e_{ij} yields $e_{ij} = e_{ji}$ so $2\mu e_{ij}[u]e_{ij}[v] = \mu e_{ij}[u](v_{i,j} + v_{j,i}) = 2\mu e_{ij}[u]v_{i,j}$. Then using $v_{i,i} = 0$ and the definition of $\tau_{ij}[u]$ we obtain $2\mu e_{ij}[u]v_{i,j} = \tau_{ij}[u]v_{i,j}$. Finally we note that $\tau_{ij}[u]v_{i,j} = \partial(\tau_{ij}[u]v_i)/\partial x_j$ because $\tau_{ij,j}[u] = 0$, as (1.3*a*) shows. Thus $2\mu e_{ij}[u]e_{ij}[v] = \partial(\tau_{ij}[u]v_i)/\partial x_j$, so the last term in (2.2) is the integral of a divergence. By Gauss' theorem we can rewrite it as a surface integral, so (2.2) becomes

$$D[u+v] = D[u] + D[v] + 2\int_{S_+ + S_-} v_i \tau_{ij}[u]n_j \, \mathrm{d}S. \tag{2.3}$$

We have used the fact that $v = 0$ on S_1.

In order for u to minimize D, the sum of the last two terms in (2.3) must be non-negative for every v satisfying (1.2*a*), (1.2*b*) and (2.1). If v satisfies these conditions, so does αv for any real number α. By replacing v by αv in (2.3), with α sufficiently small, we see that the last term in (2.3), which is linear in α, dominates $D[\alpha v]$ which is quadratic in α. Therefore the last integral must be zero, for if it were not it could be made negative by an appropriate choice of the sign of α. Thus for u to minimize D it is necessary that for all v satisfying (1.2*a*), (1.2*b*) and (2.1),

$$\int_{S_+ + S_-} v_i \tau_{ij}[u]n_j \, \mathrm{d}S = 0. \tag{2.4}$$

Now consider the v which vanish on S_-. Then (2.4) requires that the integral of $v_i\tau_{ij}[u]n_j$ over S_+ must vanish for every v for which, by (2.1), the integral of $v_i n_i$ over S_+ vanishes. This certainly will be the case if (1.3*b*) holds, for then the former integral is p_+ times the latter integral. That condition (1.3*b*) is also necessary can be seen by reformulating the requirement (2.4) as follows: the function $\tau_{ij}[u]n_j$ must be orthogonal to every function v_i which is orthogonal to n_i, so $\tau_{ij}[u]n_j$ must be a multiple of n_i, which is what (1.3*b*) states. Here orthogonality means vanishing of the inner product defined by the integral over S_+ in (2.1) and (2.4). Similarly by considering the v which vanish on S_+, we deduce (1.3*c*).

When (1.3*b*) and (1.3*c*) hold the integral in (2.3) vanishes, and (2.3) becomes $D[u+v] = D[u] + D[v]$. Since $D[v] > 0$ unless $v = 0$, it follows that the minimum of $D[u+v]$ is $D[u]$. This proves the theorem except for uniqueness. Uniqueness follows because the difference v between two solutions must vanish for both of them to minimize D, since $D(v) > 0$ unless $v = 0$. Uniqueness also follows from Theorem 5 of Keller, Rubenfeld & Molyneux (1967) in which $S_3 = S_+ + S_-$.

3. Discussion

The pressure $p(x)$ is undefined up to an additive constant which can be chosen to make $p_- = 0$ in (1.3*c*). Then in (1.3*b*) p_+ is replaced by $p_+ - p_-$. Since the problem for $u(x)$ is linear, it follows that the solution is proportional to the pressure difference $p_+ - p_-$, and therefore so is Q. Thus there is a unique pressure difference which yields a given flux Q. Since the rate of dissipation is equal to $(p_+ - p_-)Q$, minimizing the dissipation rate also minimizes the pressure drop for given flux.

Suppose that V is a channel parallel to the x_1-axis with walls at $x_2 = \pm h$ and ends $S_\pm$ at $x_1 = \pm L$. Then an admissible flow is the Poiseuille flow given by

$$u_1^P(x_2) = \frac{3Q}{4h}\left[1 - \left(\frac{x_2}{h}\right)^2\right], \quad u_2^P = 0, \quad p^P(x_1) = p_- + (x_1 + L)(p_+ - p_-)/2L. \qquad (3.1)$$

This flow satisfies (1.3*a*), but it does not satisfy (1.3*b*) or (1.3*c*). Instead it yields

$$\tau_{ij}[u^P]n_j = -p_\pm n_i \pm \frac{3\mu Q}{2h^3} x_2 \delta_{i2} \qquad \text{at } x = \pm L. \qquad (3.2)$$

In order to correct the Poiseuille flow to make it satisfy (1.3*b*) and (1.3*c*), we must add to it a flow which cancels the last term in (3.2) for $i = 2$. That term represents a shear stress in the transverse direction, proportional to x_2. The correction flow, which can be found by separation of variables, is expressible as a sum of modes. Half of the modes decay exponentially with distance from one end of the channel and the other half decay exponentially with distance from the other end. The decay rates are proportional to $1/h$, so for a long channel the correction flow is localized near the ends of the channel. The same conclusion applies to the flow in a pipe.

The preceding result shows that the minimum dissipation rate flow changes from the Poiseuille flow in the interior of the pipe or channel, to a flow with no transverse shear stress at the ends.

REFERENCES

Batchelor, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

Keller, J. B., Rubenfeld, L. A. & Molyneux, J. E. 1967 Extremum principles for slow viscous flows with applications to suspensions. *J. Fluid Mech.* **30**, 97–125.